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Separable torsion-free abelian E*-groups

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Abstract

A ring R is said to be a unique addition ring (UA-ring) if any semigroup isomorphism $R^* = (R,*) \simeq (S,*) = S^*$ of multiplicative semigroups with another ring S is always a ring isomorphism. See [5, 7–9] for earlier work on UA-rings. Depending on the context, we may or may not regard 0 as an element of R^*. An abelian group G is called a UA-group if its endomorphism ring $E(G)$ is a UA-ring. Given an abelian group G, denote by $E^*(G)$ the semigroup of all endomorphisms of G and let R_G be the collection of all rings R such that $R^* \simeq E^*(G)$. The group G is said to be an E^*-group if for every ring $(E^*(G), \oplus)$, where $\oplus$ is an addition on the semigroup $E^*(G)$, there is an abelian group H such that $(E^*(G), \oplus)$ is (isomorphic to) the endomorphism ring of H. Equivalently, G is an E^*-group if for every ring R in R_G there is an abelian group H such that R is (isomorphic to) the endomorphism ring of H.

Section 1 is a study of separable torsion-free abelian UA-groups. In Section 2 we develop necessary and sufficient conditions for a torsion-free separable group to be an E^*-group. All groups are abelian. © 1998 Elsevier Science B.V. All rights reserved.

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1. UA-groups

Lemma 1.1. *A torsion-free (abelian) group of rank one is not a UA-group.*

Proof. The endomorphism ring of a rank-one group is a subring of the rationals $\mathbb{Q}$. But no subring of the rationals is a UA-ring. Indeed, the map that interchanges two primes $p \leftrightarrow q$ in the unique factorization of an integer extends to a semigroup isomorphism from a subring of $\mathbb{Q}$ divisible by p to a subring divisible by q.

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We call a rank-one direct summand A of a torsion-free group G *semiconnected* if a complementary summand for A has a rank-one direct summand of type comparable to that of A. If every rank-one summand of G is semiconnected we call G semiconnected. The next result was announced in [6].

Theorem 1.2. *A torsion-free separable group is a UA-group if and only if it is semiconnected.*

Proof. (*only if*) Suppose that $G = A \oplus B$ with $\text{rank}(A) = 1$ and A not semiconnected. We first show that A and B are fully invariant in G. Let $\theta \in Hom(A,B)$ with $\theta(a)$ nonzero for some $a \in A$. Since B is separable, $\theta(a)$ belongs to a completely decomposable direct summand C of B. Therefore $\theta(A)$ has nonzero projection into a rank-one summand of C, whence of B. The type of this rank-one summand is then comparable to the type of A, contradicting that A is not semiconnected. We may conclude that A is fully invariant in G. Similarly, let $\theta \in Hom(B,A)$ with $\theta(b) \neq 0$ for some $b \in B$. Again using the separability of B, the element b belongs to a completely decomposable direct summand C of B. It follows that the map θ can be restricted to a nonzero map from a rank-one summand of C into A, again contradicting the fact that A is not semiconnected. Thus, B is fully invariant as well. Now we have $End(G) = End(A) \times End(B)$. By Lemma 1.1, $End(A)$ is not a UA-ring. It is immediate, therefore, that $End(A) \times End(B)$ is not a UA-ring, so that G is not a UA-group.

For the converse, we will employ the following lemma.

Lemma 1.3. *Let R be an associative ring with 1 and suppose $E = \{e_i \colon i \in I\}$ is a set of idempotents satisfying*:
1. *for any nonzero $r \in R$ there exists e_i in E such that $re_i \neq 0$;*
2. *for any idempotent $e_i \in E$ there exists an orthogonal idempotent $e_j \in E$ such that for any $x \in R$, if $e_i x e_i R e_j = 0 = e_j R e_i x e_i$, then $e_i x e_i = 0$.*

Then, R is a UA-ring.

Proof. Let $\theta : R \to S$ be a semigroup isomorphism of rings R and S. We show that θ is additive on Re_i for any $e_i \in E$, that is, $\theta(a+b) = \theta(a) + \theta(b)$ for all $a,b \in Re_i$. The proof proceeds in small steps.

First, if e_i and e_j are orthogonal idempotents of R, then (i): $\theta(e_j re_i + e_i) = \theta(e_j re_i) + \theta(e_i)$. This equality follows from the fact that both $\theta(e_i)$ and $\theta(e_j)$ left annihilate the difference $d = \theta(e_j re_i + e_i) - \theta(e_j re_i) - \theta(e_i)$, while $\theta(e_i) + \theta(e_j)$ acts like a left identity on d. Similarly, $\theta(e_j re_i + e_j) = \theta(e_j re_i) + \theta(e_j)$.

Second, (ii): $\theta(e_j re_i + e_j se_i) = \theta(e_j re_i) + \theta(e_j se_i)$. To see this, let $u = e_j re_i + e_j$ and $v = e_j se_i + e_i$. Then using (i), $\theta(uv)$ is the left side of the equality and $\theta(u)\theta(v)$ is the right. In particular, θ is additive on $(1 - e_i)Re_i$.

Third, $\theta(e_i re_i + e_i se_i) = \theta(e_i re_i) + \theta(e_i se_i)$. This equality follows from the fact that the difference $\theta(e_i re_i + e_i se_i) - \theta(e_i re_i) - \theta(e_i se_i)$ is annihilated by $\theta(Re_j)$ on the right and by $\theta(e_jR)$ on the left, for any idempotent e_j orthogonal to e_i, in view of the second equality (ii) above. In particular, we may use the idempotent e_j from hypothesis (2) of the lemma to conclude that the difference is zero.

We have shown that θ is additive on both $(1-e_i)Re_i$ and e_iRe_i. An easy calculation, analogous to those above, shows that θ is additive on Re_i as desired.

To prove that θ is a ring isomorphism, suppose that $\theta(a)+\theta(b)=\theta(c)$ and $c \neq a+b$ for some nonzero $a,b,c \in R$. By hypothesis (1), there is an idempotent $e_i \in E$ such that $ce_i \neq ae_i+be_i$. Since θ is an additive isomorphism on Re_i, we obtain the contradiction $\theta(ce_i) \neq \theta(ae_i)+\theta(be_i)$. □

We now complete the proof of Theorem 1.2 by showing that a torsion-free semiconnected separable group G satisfies the hypotheses of Lemma 1.3. For our set E of idempotents we take the projections onto rank-one summands of G. The separability of G gives immediately that if r is any endomorphism of G, then there is an idempotent $e \in E$ such that re is nonzero. That is, condition (1) of Lemma 1.3 holds. Another use of separability gives condition (2). Indeed, if e is an idempotent in E corresponding to a rank-one summand A of G, then the complementary summand for A has a rank-one summand B of type comparable to that of A. This is by the semiconnected assumption. Assume first that $type(A) < type(B)$. If e' is the idempotent projection onto the summand B, it is clear that $e'Rexe=0$ implies $exe=0$. On the other hand, if $type(B) \leq type(A)$, then $exeRe'=0$ implies $exe=0$. Thus, we may apply Lemma 1.3 to conclude that $End(G)$ is a UA-ring, so that G is a UA-group, as desired. □

2. E*-groups

Again we begin with the rank-one case.

Lemma 2.1. *Let G be a torsion-free group of rank one. Then, for any addition $\oplus$, the ring $(E^*(G),\oplus)$ is torsion-free.*

Proof. Note that for any addition $\oplus$, the additive identity is 0 and additive inverses are the usual ones because $x=-1\oplus 1$ satisfies $(-1)x=x$, so $x=0$. Since $(E^*(G),\oplus)$ is an integral domain, the characteristic of $(E^*(G),\oplus)$ is either 0 or p for some prime p. In the latter case, $(a\oplus b)^p=a^p\oplus b^p$, for all $a,b\in(E^*(G),\oplus)$. Assume $p\neq 2$, and write $2\oplus 1=a$ for some $a\in(E^*(G),\oplus)$. Then $a^{kp}=(2\oplus 1)^{kp}=2^{kp}\oplus 1$. Therefore, as endomorphisms, $a^{(k+1)p}-a^{kp}=a^{kp}(a^p-1)=2^{kp}(2^p-1)$. Since we are working with nonzero endomorphisms of a rank-one group, which can be regarded as rational numbers, the last equation implies $a=2$, leading to the contradiction $1=0$. If $p=2$, then $1\oplus 1=0$ and $1=-1$ by the uniqueness of additive inverses. This final contradiction shows that the ring $(E^*(G),\oplus)$ has characteristic zero and is therefore torsion-free. □

Lemma 2.2. *A torsion-free group of rank one is an E^*-group if and only if it is divisible by at most finitely many primes.*

Proof. Let G be torsion-free of rank one and suppose G is divisible only by a finite set P of primes. We first prove that for any addition $\oplus$, the ring $(E^*(G), \oplus)$ is reduced (as an abelian group). First suppose 1 is a divisible element of $(E^*(G), \oplus)$. A routine check shows that $\langle 1 \rangle_*$, the pure subgroup of $(E^*(G), \oplus)$ generated by 1, is in fact a subring of $(E^*(G), \oplus)$ that is isomorphic to Q. In particular, the multiplicative group $\mathbb{Q}^*$ is isomorphic to a subgroup of $E^*(G)$. But $\mathbb{Q}^* \simeq \mathbb{Z}_2 \times \bigoplus_{\aleph_0} \mathbb{Z}$ [4, p. 313]. On the other hand, the multiplicative group of $E^*(G)$ looks like $\mathbb{Z}_2 \times \bigoplus_{p \notin P} \langle p \rangle \times \bigoplus_{p \in P} \mathbb{Z}$, where $\langle p \rangle$ denotes the multiplicative semigroup generated by p. Thus, an embedding of $\mathbb{Q}^*$ into $E^*(G)$ is impossible, and 1 is not divisible, as asserted.

Next, let D be the maximal divisible subgroup of $(E^*(G), \oplus)$. There must be a nonzero integer n such that $\tilde{n} = 1 \oplus \cdots \oplus 1$ (n summands) is not invertible as an element of $(E^*(G), \oplus)$. Otherwise, the equation $\tilde{n}x = nx = x \oplus \cdots \oplus x = 1$ would always be solvable in $(E^*(G), \oplus)$ and 1 would be divisible, contradicting the first paragraph of the proof. We may assume for simplicity that n is, in fact, a prime. Suppose d_0 is a nonzero element of D. Then, d_0 may be expressed uniquely as a product of primes in the multiplicative semigroup of $E^*(G)$, regarded as a subset of Q^*. But, by divisibility, for each positive integer k there exists $d_k \in D$ such that $n^k d_k = d_0$. This contradiction shows that D must be zero and $(E^*(G), \oplus)$ is reduced. We may now apply Corner's Theorem [2] to conclude that $(E^*(G), \oplus)$ is the endomorphism ring of an abelian group, that is, G is an E^*-group.

Conversely, suppose that the group G is divisible by infinitely many primes $\{p \in P\}$. Multiplication by $p \in P$ is an automorphism of G while multiplication by a prime not in P is not an automorphism. It follows that (neglecting 0)

$$E^*(G) \simeq \mathbb{Z}_2 \times \bigoplus_{\aleph_0} \mathbb{Z} \times \bigoplus_{p \notin P} \langle p \rangle,$$

where $\mathbb{Z}_2$ is the group generated by multiplication by -1, each copy of $\mathbb{Z}$ represents the group generated by multiplication by some $p \in P$, and $\langle p \rangle$ represents the semigroup generated by multiplication by $p \notin P$ (see [4, Section XVIII]). Denote by F the field of rational functions over Q in the commuting variables $\{x_p \colon p \notin P\}$ and let A be the subring of all elements of the form f/g, where f, g are polynomials in the variables x_p such that g is nonzero whenever any x_p is set equal to 0 (x_p does not divide g). Then the nonzero multiplicative structure on A is given by

$$A^* \simeq \mathbb{Z}_2 \times \bigoplus_{\aleph_0} \mathbb{Z} \times \bigoplus_{p \notin P} \langle x_p \rangle \simeq E^*(G),$$

where each copy of Z represents the multiplicative group generated by one of the countable many invertible polynomials g and $\langle x_p \rangle$ is the multiplicative semigroup generated by x_p. Plainly, the additive group of A is a torsion-free divisible group of infinite rank. If A is the endomorphism ring of some abelian group H, then H

must be a torsion-free divisible group of infinite rank. But the endomorphism ring of such a group is uncountable. Thus, A is not an endomorphism ring and G is not an E^*-group. □

A rank-one summand of a torsion-free separable group is called *isolated* if it is not semiconnected.

Theorem 2.3. *A torsion-free separable group is an E^*-group if and only if every isolated rank-one direct summand is divisible by at most finitely many primes.*

Proof. Assume there is an isolated rank-one direct summand B of the torsion-free separable group G such that B is divisible by infinitely many primes $p \in P$. Write $G = B \oplus C$. Then B and C are fully invariant (see the proof of Theorem 1.2) so that $E(G) = E(B) \oplus E(C)$. By Lemma 2.2 there is a torsion-free ring R such that $R^* \simeq E^*(B)$, but R is not an endomorphism ring. But then $S = R \oplus E(C)$ cannot be an endomorphism ring either, and $S^* \simeq E^*(G)$. Thus, G is not an E^*-group.

Conversely, if G has no isolated direct summands, the G is a UA-group by Theorem 1.2, and therefore an E^*-group. Suppose now that G has isolated summands, all of which are divisible by at most finitely many primes. We can decompose G as $G = \bigoplus_{i \in I} G_i \oplus G'$, where each G_i is an isolated rank-one summand and G' has no isolated direct summands. This decomposition follows from the fact that if X is an isolated rank-one summand then X is the *unique* summand of type equal to *type*(X). See also [1]. Suppose that R is any ring with θ: $R^* \simeq E^*(G)$. Then, using idempotents, we can write $R = R(I) \oplus R'$, where $R' = \theta E(G')$ and $R(I) = \theta E(\bigoplus_{i \in I} G_i)$. Furthermore, for any $i \in I$, we can write $R(I) = R_i \oplus R_i'$, where $R_i = \theta E(G_i)$. Note that R_i is a ring, since $E(G_i) = e_i E(G) e_i$ for an appropriate idempotent e_i. By Lemma 2.1 and the first part of the proof of Lemma 2.2, each R_i is a reduced torsion-free ring. Since G' is a UA-group, then the isomorphism θ restricted to $E(G')$ is a ring isomorphism. Therefore R', whence R is a reduced torsion-free ring.

We next show that for any prime p, R contains no nontrivial homomorphic image of the abelian group of p-adic integers J_p. Since R is reduced, any map $J_p \to R$ must be monic (any proper homomorphic image of J_p is divisible). In particular, we can compose with projections to get maps $J_p \to R(I) \to R_i$, which must all be zero – they cannot be monic since R_i is countable. Thus, we can restrict our attention to maps $J_p \to R'$. But as noted above, G' is a UA-group. Thus, $R' \simeq E(G')$ as rings, and an embedding $\theta : J_p \to E(G')$ can be used to produce a nonzero map $\theta_g : J_p \to G'$, by $x \to \theta(x)(g)$, where $g \in G'$ is chosen to make the map nonzero. But a mapping of J_p into a reduced separable group must be zero. Otherwise, we could produce a nonzero map of J_p into a (reduced) rank one summand. Thus, R contains no nonzero homomorphic image of J_p, that is, R is cotorsion-free. By the main theorem of [3], R is the endomorphism ring of some group and G is a UA-group, as desired. □

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